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INTERPOLATION IN A CLASSICAL HILBERT SPACE OF ENTIRE FUNCTIONS[1]

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ABSTRACT. Let H denote the Paley-Wiener space of entire functions of exponential type π which belong to $L^2(-\infty,\infty)$ on the real axis. A sequence $\{\lambda_n\}$ of distinct complex numbers will be called an *interpolating sequence* for H if $TH \supset l^2$, where T is the mapping defined by $Tf = \{f(\lambda_n)\}$. If in addition $\{\lambda_n\}$ is a set of uniqueness for H, then $\{\lambda_n\}$ is called a *complete interpolating sequence*. The following results are established. If $\mathrm{Re}(\lambda_{n+1}) - \mathrm{Re}(\lambda_n) \geq \gamma > 1$ and if the imaginary part of λ_n is sufficiently small, then $\{\lambda_n\}$ is an interpolating sequence. If $|\mathrm{Re}(\lambda_n) - n| \leq L \leq (\log 2)/\pi$ $(-\infty < n < \infty)$ and if the imaginary part of λ_n is uniformly bounded, then $\{\lambda_n\}$ is a complete interpolating sequence and $\{e^{i\lambda_n t}\}$ is a basis for $L^2(-\pi,\pi)$. These results are used to investigate interpolating sequences in several related spaces of entire functions of exponential type.

Introduction. Let H denote the Paley-Wiener space of entire functions f of exponential type π for which

$$\|f\| = \left[\int_{-\infty}^{\infty} |f(x)|^2\,dx\right]^{1/2} < \infty.$$

A sequence $\{\lambda_n\}$ of distinct complex numbers will be called an *interpolating sequence* for H if corresponding to each sequence $\{c_n\}$ in l^2 there is at least one function f in H satisfying $f(\lambda_n) = c_n$ $(-\infty < n < \infty)$. (Unless otherwise stated, the term *sequence* in this paper will always mean a *two-sided* sequence.)

In §§2 and 3 of this paper we study both necessary and sufficient conditions for a sequence $\{\lambda_n\}$ to be interpolating. For the most part we require that the λ_n lie in a strip parallel to the real axis. Under this condition we show that l^2 is the *natural* sequence space to interpolate with functions in H and a necessary condition for interpolation is that the λ_n be separated, that is, $|\lambda_n - \lambda_k| \geq \gamma$ for some constant $\gamma > 0$ and all $n \neq k$.

A classical theorem of Paley and Wiener [8, p. 13] shows that the Fourier transform of every function in H vanishes almost everywhere outside $(-\pi,\pi)$. Thus f belongs to H if and only if it is of the form

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{izt}\,dt \tag{1}$$

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for some g in $L^2(-\pi,\pi)$. It follows that $\{\lambda_n\}$ is an interpolating sequence if and only if the trigonometric moment problem

$$(2) \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{i\lambda_n t}\,dt \qquad (-\infty < n < \infty)$$

has a solution g in $L^2(-\pi,\pi)$ whenever $\sum |c_n|^2 < \infty$. The following result was first established by Boas [3] and later reproved in a more abstract setting by N. Bari [1]. (See also [10].)

Lemma 1. *A necessary and sufficient condition that the trigonometric moment problem* (2) *have a solution in* $L^2(-\pi,\pi)$ *for every square summable sequence* $\{c_n\}$ *is that the inequality*

$$(3) \qquad A\sum |a_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum a_n e^{i\lambda_n t}\right|^2 dt$$

holds for some constant $A > 0$ *and all finite sequences* $\{a_n\}$.

Several years earlier, Ingham [7] had established the validity of (3) in the special case in which the λ_n are real and satisfy the separation condition $\lambda_{n+1} - \lambda_n \geq \gamma > 1$ $(-\infty < n < \infty)$. He showed that his result is the best possible, in the sense that γ cannot be taken equal to 1, by demonstrating that the sequence $\{\lambda_n\}$ given by

$$\lambda_n = n - \frac{1}{4}, \qquad \lambda_{-n} = -\lambda_n \qquad (n = 1,2,\dots)$$

does not satisfy (3) for any positive A. By modifying Ingham's proof, we are able to show that $\{\lambda_n\}$ is an interpolating sequence whenever

$$\mathrm{Re}(\lambda_{n+1}) - \mathrm{Re}(\lambda_n) \geq \gamma > 1 \qquad (-\infty < n < \infty)$$

and the imaginary part of λ_n is sufficiently small. As a corollary we show that if $\{\lambda_n\}$ is a sequence of points lying in a strip parallel to the real axis and if $\mathrm{Re}(\lambda_{n+1}) - \mathrm{Re}(\lambda_n) \to \infty$ as $n \to \pm\infty$, then for each $\{c_n\}$ in l^2 and each $\tau > 0$ there exists an entire function f of exponential type τ, square integrable on the real axis, for which $f(\lambda_n) = c_n$ $(-\infty < n < \infty)$.

In their classic treatise, Paley and Wiener showed [8, p.115] that every sequence of real numbers $\{\lambda_n\}$ which are close to the integers in the sense that $|\lambda_n - n| \leq L < 1/\pi^2$ $(-\infty < n < \infty)$ is an interpolating sequence. Moreover, they showed that every function g in $L^2(-\pi,\pi)$ has a *nonharmonic* Fourier series expansion

$$(4) \qquad g(t) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{-N}^{N} c_n e^{i\lambda_n t}$$

with $\sum |c_n|^2 < \infty$. These results were improved by Duffin and Eachus [5] who showed that λ_n can be complex and that the constant $1/\pi^2$ can be replaced by $(\log 2)/\pi$. In the present paper we show that $\{\lambda_n\}$ is an interpolating sequence whenever $|\mathrm{Re}(\lambda_n) - n| \le L < (\log 2)/\pi$ $(-\infty < n < \infty)$ and the imaginary part of λ_n is uniformly bounded, and that under these conditions every function in $L^2(-\pi, \pi)$ has the representation (4) with some sequence $\{c_n\}$ in l^2.

The main results of §§2 and 3 are used in §4 to investigate interpolating sequences in several related spaces of entire functions of exponential type. Specifically, we consider the spaces E_τ^p of entire functions of exponential type τ which belong to $L^p(-\infty, \infty)$ on the real axis. We restrict attention mainly to the special cases $p = 1$ and $p = \infty$.

1. **Background material.** Under the inner product

$$(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx,$$

H is a functional Hilbert space with the reproducing kernel

$$K(\xi, z) = \sin \pi(\xi - \bar{z})/\pi(\xi - \bar{z}).$$

The functions $K(\xi, n)$, $-\infty < n < \infty$, form a complete orthonormal system, so that for each f in H

$$f(z) = \sum_{-\infty}^{\infty} f(n) \frac{\sin \pi(z - n)}{\pi(z - n)}.$$

By Parseval's formula

$$\|f\|^2 = \sum_{-\infty}^{\infty} |f(n)|^2 \qquad (f \in H). \tag{5}$$

If f is given by (1), then g is the Fourier transform of f and Plancherel's theorem gives

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx. \tag{6}$$

It follows easily from (1) and (6) that

$$|f(x + iy)| \le e^{\pi|y|}\|f\| \tag{7}$$

for every f belonging to H. Similar estimates show that H is closed under differentiation and that

$$\|f'\| \le \pi\|f\| \qquad (f \in H). \tag{8}$$

2. **Interpolation in H.** Let $\{\lambda_n\}$ be a sequence of distinct complex numbers. For each f in H we define Tf to be the sequence $Tf = \{f(\lambda_n)\}$ $(-\infty < n < \infty)$.

Definition. The sequence $\{\lambda_n\}$ is said to be an *interpolating sequence* for H if $TH \supset l^2$. It is not difficult to show that if $TH \supset l^2$, then *the unit ball in l^2 can be interpolated in a uniformly bounded way.* In fact, we have the following stronger result. For a proof see [9, p. 19].

Lemma 2. *Let X be a Banach space, $\{\mu_n\}$ a sequence in the dual space X' and T the mapping on X defined by $Tx = \{\mu_n(x)\}$. If $TX \supset l^p$ for some p, $1 \leq p \leq \infty$, then TM covers the unit ball of l^p for some bounded subset M of X.*

Definition. The sequence $\{\lambda_n\}$ is said to be *separated* if there is a constant γ such that $|\lambda_n - \lambda_k| \geq \gamma > 0$ $(n \neq k)$.

It is well known [4, p. 101] that if the λ_n are *real* and separated and if f is an entire function of exponential type τ belonging to $L^p(-\infty, \infty)$ on the real axis $(0 < p < \infty)$, then

$$\sum |f(\lambda_n)|^p \leq B \int_{-\infty}^{\infty} |f(x)|^p \, dx,$$

where B depends only on p, τ and the separation constant γ. It is no more difficult to establish the following result, which we state without proof.

Lemma 3. *Let $\{\lambda_n\}$ be a complex sequence satisfying*

$$|\mathrm{Im}(\lambda_n)| \leq \alpha, \qquad |\lambda_n - \lambda_k| \geq \gamma \qquad (n \neq k)$$

for some positive constants α and γ. If f is an entire function of exponential type τ, belonging to $L^p(-\infty, \infty)$ on the real axis, then

$$\sum |f(\lambda_n)|^p \leq B \int_{-\infty}^{\infty} |f(x)|^p \, dx,$$

where $B = B(p, \tau, \gamma, \alpha)$.

Proposition 1. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TH \supset l^1$, then $\{\lambda_n\}$ is separated.*

Proof. Since point-evaluations are continuous linear functionals on H, Lemma 2 shows that there are functions f_n in H and a constant $M > 0$ such that $f_n(\lambda_k) = \delta_{nk}$, $\|f_n\| \leq M$ (for all n, k). For $n \neq k$, we have

$$1 = f_n(\lambda_n) - f_n(\lambda_k) = \int_{\lambda_k}^{\lambda_n} f_n'(z) \, dz$$

so that

$$1 \leq \int_{\lambda_k}^{\lambda_n} |f_n'(z)| \, |dz| \leq \sup |f_n'(z)| \cdot |\lambda_n - \lambda_k|,$$

where the supremum is taken over the straight line segment from λ_k to λ_n. If $|\mathrm{Im}(\lambda_n)| \leq \alpha$, then (7), (8) show that

$$\begin{aligned} 1 &\leq e^{\pi\alpha} \|f_n'\| \cdot |\lambda_n - \lambda_k| \\ &\leq \pi e^{\pi\alpha} \|f_n\| \cdot |\lambda_n - \lambda_k| \\ &\leq M\pi e^{\pi\alpha} |\lambda_n - \lambda_k|, \end{aligned}$$

and the result follows with $\gamma = e^{-\pi\alpha}/M\pi$.

Corollary 1. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TH \supset l^1$, then $TH \subset l^2$.*

Since separation is a necessary condition for a sequence of real numbers to be interpolating, the problem arises to determine if there is a constant γ with the property that $\{\lambda_n\}$ is an interpolating sequence whenever $\lambda_{n+1} - \lambda_n \geq \gamma$. The following result will show that $\{\lambda_n\}$ is interpolating whenever $\gamma > 1$. The proof is a simple extension of an argument given by Ingham [7].

Lemma 4. *Let $f(t) = \sum_{-N}^{N} a_n e^{i\lambda_n t}$ where the λ_n satisfy*

$$\text{(9)} \qquad \operatorname{Re}(\lambda_{n+1}) - \operatorname{Re}(\lambda_n) \geq \gamma > 1, \qquad |\operatorname{Im}(\lambda_n)| \leq \alpha.$$

Then

$$A \sum |a_n|^2 \leq \int_{-\pi}^{\pi} |f(t)|^2 \, dt,$$

where $A = 4[1/(1 + 16\alpha^2) - e^{2\pi\alpha}/\gamma^2]$.

Proof. If k belongs to $L^1(-\infty, \infty)$ and $K(z) = \int_{-\infty}^{\infty} k(t) e^{izt} \, dt$, then

$$\text{(10)} \qquad \int_{-\infty}^{\infty} k(t) |f(t)|^2 \, dt = \sum_{m,n} a_m \bar{a}_n K(\lambda_m - \bar{\lambda}_n).$$

Letting

$$\begin{aligned} k(t) &= \cos \tfrac{1}{2}t, \qquad |t| \leq \pi, \\ &= 0, \qquad |t| > \pi, \end{aligned}$$

we get $K(z) = 4(\cos \pi z)/(1 - 4z^2)$. Since $|k(t)| \leq 1$, it follows from (10) that

$$\int_{-\pi}^{\pi} |f(t)|^2 \, dt \geq \left| \sum_n |a_n|^2 K(\lambda_n - \bar{\lambda}_n) \right| - \left| \sum_{m,n}{}' a_m \bar{a}_n K(\lambda_m - \bar{\lambda}_n) \right|,$$

where the prime in the summation denotes omission of the term $m = n$. The remainder of the proof is devoted to obtaining suitable estimates for the two sums in absolute value above. Since, by (9), $|\lambda_m - \bar{\lambda}_n| \geq |m - n|\gamma > 1$ $(m \neq n)$ and since $|\cos(x + iy)| \leq e^{|y|}$, we have

$$\begin{aligned} \sum_m{}' |K(\lambda_m - \bar{\lambda}_n)| &\leq 4e^{2\pi\alpha} \sum_m{}' \frac{1}{4(m - n)^2\gamma^2 - 1} \\ &< \frac{8e^{2\pi\alpha}}{\gamma^2} \sum_{r=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{4e^{2\pi\alpha}}{\gamma^2}. \end{aligned}$$

Since $2|a_m a_n| \leq |a_m|^2 + |a_n|^2$, we get

$$\sum_{m,n}{}' a_m \bar{a}_n K(\lambda_m - \bar{\lambda}_n) = \theta \sum_{m,n}{}' \frac{|a_m|^2 + |a_n|^2}{2} |K(\lambda_m - \bar{\lambda}_n)|,$$

where $|\theta| \leq 1$, and since $|K(z)| = |K(-\bar{z})|$,

$$\sum_{m,n}' a_m \bar{a}_n K(\lambda_m - \bar{\lambda}_n) = \theta \sum_n |a_n|^2 \left\{ \sum_m' |K(\lambda_m - \bar{\lambda}_n)| \right\}$$

$$< \frac{4e^{2\pi\alpha}}{\gamma^2} \sum_n |a_n|^2.$$

If we set $\beta_n = \operatorname{Im}(\lambda_n)$, then

$$K(2i\beta_n) = \frac{4\cosh 2\pi\beta_n}{1 + 16\beta_n^2} \geq \frac{4}{1 + 16\alpha^2},$$

and hence

$$\sum_n |a_n|^2 K(\lambda_n - \bar{\lambda}_n) \geq \frac{4}{1 + 16\alpha^2} \sum |a_n|^2,$$

and the proof is complete.

Theorem 1. *Let* $\{\lambda_n\}$ *be a complex sequence satsifying*

$$\operatorname{Re}(\lambda_{n+1}) - \operatorname{Re}(\lambda_n) \geq \gamma > 1,$$
$$|\operatorname{Im}(\lambda_n)| \leq \alpha \qquad (-\infty < n < \infty).$$

Then $\{\lambda_n\}$ *is an interpolating sequence for all sufficiently small values of* α.

Proof. The result follows immediately from Lemmas 1 and 4 since the value of A given in the statement of Lemma 4 approaches $4(1 - 1/\gamma^2) > 0$ as $\alpha \to 0$.

Remark. Shapiro and Shields have shown [10, p. 532] that if $\{\lambda_n\}$ is a separated sequence of real numbers and if $\sum |c_n|^2 < \infty$, then for each $\alpha > 0$ there corresponds a function f analytic in the strip D: $|y| < \alpha$, with finite area norm

$$\iint_D |f(z)|^2 \, dx\, dy < \infty, \tag{11}$$

such that $f(\lambda_n) = c_n$ $(-\infty < n < \infty)$. Since (11) is satisfied for each function in H, an obvious extension of Theorem 1 shows that f may in fact be chosen to be entire of exponential type.

Theorem 2. *Let* $\{\lambda_n\}$ *be a sequence of distinct points lying in a strip parallel to the real axis and suppose that* $\operatorname{Re}(\lambda_{n+1}) - \operatorname{Re}(\lambda_n) \to \infty$, $n \to \pm\infty$. *For each* $\tau > 0$ *and each sequence* $\{c_n\}$ *in* l^2 *there exists an entire function* f *of exponential type* τ, *belonging to* $L^2(-\infty, \infty)$ *on the real axis, for which* $f(\lambda_n) = c_n$ $(-\infty < n < \infty)$.

Proof. Fix $\tau > 0$ and let H_τ denote the space of entire functions of exponential type τ which belong to $L^2(-\infty, \infty)$ on the real axis. It follows from Theorem 1 that there exists a smallest integer $N \geq 0$ with the property that for each sequence $\{c_n\}$ in l^2 there corresponds at least one function f in H_τ such that $f(\lambda_n) = c_n$, $|n| \geq N$. If $N = 0$ there is nothing to show, so suppose that $N \geq 1$. Choose g in H_τ such that

$$g(\lambda_N) = 1, \qquad g(\lambda_n) = 0, \qquad |n| \geq N, n \neq N.$$

We begin by constructing a function h in H_τ such that

$$h(\lambda_{N-1}) = 1, \qquad h(\lambda_n) = 0, \qquad |n| \geq N. \tag{12}$$

If $g(\lambda_{N-1}) = 0$, then for suitable constants c and $m \geq 1$ the function

$$h(z) = \left[\frac{g(z)}{c(z - \lambda_{N-1})^m}\right]\left[\frac{z - \lambda_N}{\lambda_{N-1} - \lambda_N}\right]$$

satisfies (12). If $g(\lambda_{N-1}) \neq 0$, then h must be obtained by a different method. By a theorem of Titchmarsh [11] the zeros of g have a positive density, where it is understood that multiple zeros are counted as many times as their multiplicity warrants. It is easily shown that $\{\lambda_n\}$ has density zero, so that some zero λ of g is either not in the sequence or else is in the sequence and is a multiple zero of g. In either case, the function

$$h(z) = \frac{\lambda_{N-1} - \lambda}{\lambda_{N-1} - \lambda_N}\left[\frac{g(z)}{g(\lambda_{N-1})} \cdot \frac{z - \lambda_N}{z - \lambda}\right]$$

satisfies (12).

Now fix $\{c_n\}$ in l^2 and choose f in H_τ such that $f(\lambda_n) = c_n$ ($|n| \geq N$). Let g and h be chosen as above and define

$$f_1(z) = f(z) + [c_{N-1} - f(\lambda_{N-1})]h(z).$$

Then f_1 belongs to H_τ and $f_1(\lambda_n) = c_n$ ($|n| \geq N$ and $n = N - 1$). The same construction gives $f_2 \in H_\tau$ such that $f_2(\lambda_n) = c_n$ ($|n| \geq N - 1$). But this contradicts the choice of N; hence N must be equal to zero, and the proof is complete.

Theorem 3. *Let $\{\lambda_n\}$, $n = 1, 2, \ldots,$ be an interpolating sequence. There exist positive numbers δ_n such that $\{\mu_n\}$ is an interpolating sequence whenever $|\lambda_n - \mu_n| < \delta_n$.*

Proof. It follows from Lemma 2 that there is a constant $M > 0$ depending only on $\{\lambda_n\}$ with the property that for each sequence $\{c_n\}$ in the unit ball of l^2 there corresponds at least one function f in H for which $f(\lambda_n) = c_n$ $(n = 1, 2, \ldots)$ and $\|f\| \leq M$.

Let $\mathfrak{F}_1$ denote the family of all functions f in H for which $\|f\| \leq M$. Then (7) shows that the functions in $\mathfrak{F}_1$ are uniformly bounded on compacta. It follows that $\mathfrak{F}_1$ is equicontinuous on compacta and hence in particular on the disk $|z - \lambda_1| \leq 1$. Thus, if $0 < \varepsilon_1 < 1$, there is a corresponding $\delta_1 = \delta_1(\varepsilon_1)$ such that $|f(z) - f(\lambda_1)| < \varepsilon$ $(|z - \lambda_1| < \delta_1)$, uniformly for all f in $\mathfrak{F}_1$. In addition, δ_1 may be chosen small enough so that the disk $|z - \lambda_1| < \delta_1$ intersects $\{\lambda_n\}$ only at λ_1.

We are going to show that if $|\lambda_1 - \mu_1| < \delta_1$, then $\{\mu_1, \lambda_2, \lambda_3, \ldots\}$ is an interpolating sequence. Clearly, it will be enough to show that the unit ball of l^2 can be interpolated. More precisely, we will show that for each $\{c_n\}$ with $\sum |c_n|^2 \leq 1$ there corresponds a function F_1 in H for which

$$F_1(\mu_1) = c_1,$$
$$F_1(\lambda_n) = c_n, \qquad n = 2, 3, \ldots.$$
$$\|F_1\| \leq M/(1 - \varepsilon_1),$$

Fix μ_1 with $|\lambda_1 - \mu_1| < \delta_1$ and let $\sum |c_n|^2 \leq 1$. There exists a function g in H such that

$$g(\lambda_n) = c_n, \qquad \|g\| \leq M, \qquad n = 1, 2, \ldots.$$

Since g is in $\mathfrak{F}_1$, $|g(\mu_1) - g(\lambda_1)| < \varepsilon_1$. Also, there exists a function f in H such that

$$f(\lambda_1) = 1,$$
$$f(\lambda_n) = 0, \qquad n = 2, 3, \ldots.$$
$$\|f\| \leq M,$$

Then f is also in $\mathfrak{F}_1$ so that $|f(\mu_1) - f(\lambda_1)| < \varepsilon_1$, and hence $|f(\mu_1)| > 1 - \varepsilon_1 > 0$. Now, set

$$F_1(z) = g(z) + [c_1 - g(\mu_1)]f(z)/f(\mu_1).$$

Clearly, F_1 is in H, $F_1(\mu_1) = c_1$, $F_1(\lambda_n) = c_n$ $(n > 1)$, and

$$\begin{aligned} \|F_1\| &\leq \|g\| + \frac{|c_1 - g(\mu_1)|}{|f(\mu_1)|}\|f\| \\ &\leq \|g\| + \frac{|g(\lambda_1) - g(\mu_1)|}{|f(\mu_1)|}\|f\| \\ &\leq M(1 + \varepsilon_1/(1 - \varepsilon_1)) = M/(1 - \varepsilon_1). \end{aligned}$$

The above argument can be repeated with $\{\lambda_n\}$ replaced by $\{\mu_1, \lambda_2, \lambda_3, \ldots\}$. Thus, we let $\mathfrak{F}_2$ denote the family of all functions f in H for which $\|f\| \leq M/(1 - \varepsilon_1)$. Then $\mathfrak{F}_2$ is equicontinuous on each compact set and for $0 < \varepsilon_2 < 1$ we find $\delta_2 = \delta_2(\varepsilon_1, \varepsilon_2)$ such that

$$|f(z) - f(\lambda_2)| < \varepsilon_2 \qquad (|z - \lambda_2| < \delta_2),$$

uniformly for all f in $\mathfrak{F}_2$. We note that δ_2 is independent of μ_1 and may be chosen so that the disks $|z - \lambda_1| < \delta_1$ and $|z - \lambda_2| < \delta_2$ are disjoint and intersect $\{\lambda_n\}$ only at λ_1 and λ_2, respectively. Just as before, we show that whenever $|\mu_2 - \lambda_2| < \delta_2$, the sequence $\{\mu_1, \mu_2, \lambda_3, \lambda_4, \ldots\}$ is interpolating, and that for each sequence $\{c_n\}$ in the unit ball of l^2 there corresponds a function F_2 in H for which

$$F_2(\mu_n) = c_n, \qquad n = 1, 2,$$
$$F_2(\lambda_n) = c_n, \qquad n > 2,$$
$$\|F_2\| \leq M/(1 - \varepsilon_1)(1 - \varepsilon_2).$$

The above process may be iterated. Thus, given a sequence $\{\varepsilon_n\}$, $0 < \varepsilon_n < 1$, we obtain a corresponding sequence $\{\delta_n\}$, $\delta_n = \delta_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) > 0$, with the property that for each positive integer N the sequence $\{\mu_1, \mu_2, \ldots, \mu_N, \lambda_{N+1}, \lambda_{N+2}, \ldots\}$ is interpolating whenever $|\lambda_n - \mu_n| < \delta_n$ ($n = 1, 2, \ldots, N$), and such that for every sequence $\{c_n\}$ with $\sum |c_n|^2 \leq 1$ there exists a function F_N in H for which

$$F_N(\mu_n) = c_n, \qquad n = 1, 2, \ldots, N,$$
$$F_N(\lambda_n) = c_n, \qquad n > N,$$
$$\|F_N\| \leq M \frac{1}{1 - \varepsilon_1} \frac{1}{1 - \varepsilon_2} \cdots \frac{1}{1 - \varepsilon_N}.$$

Now, choose $\{\varepsilon_n\}$, $0 < \varepsilon_n < 1$, so that

$$\varepsilon = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{1 - \varepsilon_n} < \infty,$$

and let the corresponding sequence $\{\delta_n\}$ be determined as above. Fix $\{\mu_n\}$ with $|\lambda_n - \mu_n| < \delta_n$ ($n = 1, 2, \ldots$) and let $\{c_n\}$ belong to the unit ball of l^2. For each positive integer N there exists a function F_N in H such that

$$F_N(\mu_n) = c_n, \qquad n = 1, 2, \ldots, N,$$
$$\|F_N\| \leq M \prod_{n=1}^{N} \left(1 + \frac{\varepsilon_n}{1 - \varepsilon_n}\right)$$
$$\leq M \exp\left[\sum_{n=1}^{\infty} \frac{\varepsilon_n}{1 - \varepsilon_n}\right] = Me^{\varepsilon}.$$

Since $\|F_N\|$ is uniformly bounded, a subsequence of $\{F_N\}$ will converge weakly to a function F in H for which $F(\mu_n) = c_n$ ($n = 1, 2, \ldots$). Thus, the unit ball of l^2 can be interpolated and the proof is complete.

3. Uniqueness: complete interpolatng sequences.

Proposition 2. *Let* $\{\lambda_n\}$, $n = 1, 2, \ldots$, *be an interpolating sequence. Each of the following statements implies the others.*

(i) *The set of relations* $f \in H$ *and* $f(\lambda_n) = 0$ ($n = 1, 2, \ldots$) *imply that* $f \equiv 0$, *that is,* $\{\lambda_n\}$ *is a set of uniqueness for* H.

(ii) *The exponentials* $\{e^{i\lambda_n t}\}$ *are complete in* $L^2(-\pi, \pi)$.

(iii) *The sequence* $\{\lambda_n\}$ *is contained in no larger interpolating sequence.*

Proof. It follows immediately from the Paley-Wiener representation (1) that (i) and (ii) are equivalent. Since (ii) clearly implies (iii) it remains only to show that (iii) implies (ii).

Suppose then that $\{e^{i\lambda_n t}\}$ is not complete in $L^2(-\pi,\pi)$ and let λ_0 be distinct from the λ_n. We show that $\{\lambda_n\}$, $n = 0, 1, 2, \ldots$, is an interpolating sequence. Since $\{e^{i\lambda_n t}\}$ is incomplete there is a function g in $L^2(-\pi,\pi)$, $g \not\equiv 0$, such that

$$\int_{-\pi}^{\pi} g(t)e^{i\lambda_n t}\,dt = 0, \qquad n = 1, 2, \ldots.$$

Setting $h(z) = \int_{-\pi}^{\pi} g(t)e^{izt}\,dt$, we have $h \in H$, $h \not\equiv 0$, and $h(\lambda_n) = 0$ ($n = 1, 2, \ldots$). If $h(\lambda_0) \neq 0$, we set $F(z) = h(z)/h(\lambda_0)$, while if $h(\lambda_0) = 0$, we take $F(z) = h(z)/A(z-\lambda_0)^m$, where A and m are chosen so that $F(\lambda_0) = 1$. In either case, $F(\lambda_0) = 1$ and $F(\lambda_n) = 0$ ($n = 1, 2, \ldots$).

Fix $\{c_n\}$, $n = 0, 1, 2, \ldots$, in l^2. There is a function G in H with $G(\lambda_n) = c_n$ ($n = 1, 2, \ldots$). Let

$$f(z) = G(z) + [c_0 - G(\lambda_0)]F(z).$$

Then f is in H and $f(\lambda_n) = c_n$ ($n = 0, 1, 2, \ldots$).

Definition. An interpolating sequence satisfying any one of the conditions listed in Proposition 2 will be called a *complete interpolating sequence.*

Theorem 4. *Let $\{\lambda_n\}$ be a sequence of distinct points lying in a strip parallel to the real axis. If $\{\mathrm{Re}(\lambda_n)\}$ is a complete interpolating sequence, then $\{\lambda_n\}$ is a complete interpolating sequence.*

The proof of Theorem 4 requires the following lemma.

Lemma 5. *Let $\lambda_n = \alpha_n + i\beta_n$, where α_n and β_n are real and satisfy*

$$\alpha_{n+1} - \alpha_n \geq \gamma > 0, \qquad |\beta_n| \leq \beta, \qquad -\infty < n < \infty.$$

If $\{e^{i\lambda_n t}\}$ is complete in $L^2(-\pi,\pi)$, then $\{e^{i\alpha_n t}\}$ is also complete in $L^2(-\pi,\pi)$.

Proof of Lemma 5. An equivalent problem is to show that the completeness of $\{e^{i(\lambda_n+i)t}\}$ implies that of $\{e^{i(\alpha_n+i)t}\}$. For this it is enough to show that the only function in H which vanishes at every point $\alpha_n + i$ is identically zero. Arguing by contradiction, we suppose that for some f in H, $f \not\equiv 0$, $f(\alpha_n + i) = 0$ ($-\infty < n < \infty$). Without any loss of generality we may suppose that no α_n is an integer and that $f(0) = 1$. We are going to exhibit a function g in H with $g(\lambda_n + i) = 0$ ($-\infty < n < \infty$) and $g(0) = 1$, thereby contradicting the completeness of $\{e^{i(\lambda_n+i)t}\}$. Set

$$f_N(z) = f(z)\prod_{n=-N}^{N} \frac{1 - z/(\lambda_n + i)}{1 - z/(\alpha_n + i)}, \qquad N = 1, 2, \ldots.$$

For each N we have $f_N \in H$, $f_N(\lambda_n + i) = 0$ ($|n| \leq N$), and $f_N(0) = 1$. By (5),

$$\|f_N\|^2 = \sum_{k=-\infty}^{\infty} |f_N(k)|^2 = \sum_{k=-\infty}^{\infty} |f(k)|^2 \left[\prod_{n=-N}^{N} \left| \frac{1 - k/(\lambda_n + i)}{1 - k/(\alpha_n + i)} \right|^2 \right].$$

We show that the products

$$\prod_{n=-N}^{N} \left| \frac{1 - k/(\lambda_n + i)}{1 - k/(\alpha_n + i)} \right|^2 = \prod_{n=-N}^{N} \left| \frac{\alpha_n + i}{\lambda_n + i} \right|^2 \left| \frac{\lambda_n + i - k}{\alpha_n + i - k} \right|^2$$

are uniformly bounded in N and k. Simple calculations show that

$$\left| \frac{\alpha_n + i}{\lambda_n + i} \right|^2 = \frac{\alpha_n^2 + 1}{\alpha_n^2 + (\beta_n + 1)^2} = 1 - \frac{\beta_n^2 + 2\beta_n}{\alpha_n^2 + (\beta_n + 1)^2} \leq 1 + \frac{\beta_n^2 + 2|\beta_n|}{\alpha_n^2 + (\beta_n + 1)^2}$$

$$\leq 1 + \frac{\beta^2 + 2\beta}{\alpha_n^2} = 1 + \frac{A}{\alpha_n^2},$$

and

$$\left| \frac{\lambda_n + i - k}{\alpha_n + i - k} \right|^2 = \frac{(\alpha_n - k)^2 + (\beta_n + 1)^2}{(\alpha_n - k)^2 + 1} \leq \frac{(\alpha_n - k)^2 + (\beta + 1)^2}{(\alpha_n - k)^2 + 1}$$

$$= 1 + \frac{\beta^2 + 2\beta}{(\alpha_n - k)^2 + 1} = 1 + \frac{B}{(\alpha_n - k)^2 + 1}.$$

Therefore

$$\prod_{n=-N}^{N} \left| \frac{1 - k/(\lambda_n + i)}{1 - k/(\alpha_n + i)} \right|^2 \leq \prod_{n=-\infty}^{\infty} \left[1 + \frac{A}{\alpha_n^2} \right] \left[1 + \frac{B}{(\alpha_n - k)^2 + 1} \right]$$

$$\leq \exp\left\{ \sum_{n=-\infty}^{\infty} \left[\frac{A}{\alpha_n^2} + \frac{B}{(\alpha_n - k)^2 + 1} \right] \right\}.$$

Since $\{\alpha_n\}$ is separated and no α_n vanishes, the series $\sum \alpha_n^{-2}$ converges and

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\alpha_n - k)^2 + 1} \leq 2 + \sum_{n=1}^{\infty} \frac{1}{(n\gamma)^2} \qquad (-\infty < k < \infty).$$

It follows that $\sup \|f_N\| < \infty$, so that a subsequence of $\{f_N\}$ converges weakly to a function g in H for which $g(\lambda_n + i) = 0$ $(-\infty < n < \infty)$ and $g(0) = 1$.

Proof of Theorem 4. Let $\lambda_n = \alpha_n + i\beta_n$. Since $\{\alpha_n\}$ is a complete interpolating sequence, it follows from Lemma 3 and Proposition 1 that the mapping $T\colon H \to l^2$ given by $Tf = \{f(\alpha_n)\}$ is continuous, one-to-one, and onto. By the open mapping theorem, T has a continuous inverse. Thus, there exists a positive constant A such that

$$A\|f\|^2 \leq \sum |f(\alpha_n)|^2 \qquad (f \in H).$$

By a theorem of Duffin and Schaeffer [6, p. 355] we have

$$B\|f\|^2 \leq \sum |f(\lambda_n)|^2 \tag{13}$$

for some constant $B > 0$ and all f in H. Now there exist functions g_n in H such that $g_n(\alpha_k) = \delta_{nk}$. It follows from Lemma 5 that for each k the sequence of functions $\{e^{i\lambda_n t}\}_{n\neq k}$ is incomplete in $L^2(-\pi,\pi)$. Therefore, we can find functions f_n in H such that $f_n(\lambda_k) = \delta_{nk}$. Fix $\{c_n\}$ in l^2 and set $F_N(z) = \sum_{-N}^{N} c_n f_n(z)$ $(N = 1,2,\dots)$. Since $F_N(\lambda_k)$ is equal to c_k when $|k| \leq N$ and has the value 0 for $|k| > N$, (13) gives

$$\|F_N\|^2 \leq \frac{1}{B}\sum_{k=-\infty}^{\infty} |F_N(\lambda_k)|^2 \leq \frac{1}{B}\sum_{-\infty}^{\infty} |c_k|^2$$

so that a subsequence of $\{F_N\}$ converges weakly to a function $F \in H$ for which $F(\lambda_k) = c_k$ $(-\infty < k < \infty)$.

Corollary 2. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis, and suppose that*

$$|\mathrm{Re}(\lambda_n) - n| \leq L < (\log 2)/\pi \qquad (-\infty < n < \infty).$$

Then $\{\lambda_n\}$ is a complete interpolating sequence.

Proof. It was shown by Duffin and Eachus [5] that the inequality

$$\|\sum c_n(e^{i\mathrm{Re}(\lambda_n)t} - e^{int})\|^2_{L^2(-\pi,\pi)} \leq \theta^2 \sum |c_n|^2 \tag{14}$$

holds for some constant θ, $0 \leq \theta < 1$, and every sequence $\{c_n\}$ in l^2. By a theorem of Paley and Wiener [8, p. 100], $\{\mathrm{Re}(\lambda_n)\}$ is a complete interpolating sequence. The result then follows from Theorem 4.

Theorem 5. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $\{\lambda_n\}$ is a complete interpolating sequence, then each function in $L^2(-\pi,\pi)$ has a unique expansion of the form*

$$g(t) = \underset{N\to\infty}{\mathrm{l.i.m.}} \sum_{-N}^{N} c_n e^{i\lambda_n t}.$$

Moreover, $A\sum |c_n|^2 \leq \|g\|^2_{L^2(-\pi,\pi)} \leq B\sum |c_n|^2$, where A and B are positive constants independent of f.

Proof. It follows from Lemma 1 that the inequality

$$A\sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|^2_{L^2(-\pi,\pi)}$$

holds for some $A > 0$ and all finite sequences $\{c_n\}$. Since $\{\lambda_n\}$ is separated (Proposition 1), Lemma 3 shows that

$$\sum |f(\lambda_n)|^2 \leq B\|f\|^2 \tag{15}$$

for some $B > 0$ and every f in H.

If K_n denotes the reproducing function at λ_n,

$$K_n(z) = K(z,\lambda_n) = \frac{\sin \pi(z - \bar{\lambda}_n)}{\pi(z - \bar{\lambda}_n)},$$

then (15) may be rewritten as

$$\sum |(f, K_n)|^2 \leq B\|f\|^2. \tag{16}$$

Let us set $f = \sum c_n K_n$ where $\{c_n\}$ is a finite sequence. By (16) and the Cauchy-Schwarz inequality

$$\begin{aligned}\|f\|^2 &= (f, \sum c_n K_n) = \sum \bar{c}_n (f, K_n) \\ &\leq [\sum |c_n|^2]^{1/2} [\sum |(f, K_n)|^2]^{1/2} \\ &\leq |\sum |c_n|^2|^{1/2} B^{1/2} \|f\|,\end{aligned}$$

so that $\|f\|^2 \leq B \sum |c_n|^2$. Taking the Fourier transform of f we get

$$A \sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|^2_{L^2(-\pi,\pi)} \leq B \sum |c_n|^2 \tag{17}$$

for every finite sequence $\{c_n\}$ and hence for every sequence in l^2. It is a simple consequence of (17) that each function in $L^2(-\pi, \pi)$ which lies in the closed linear span of $\{e^{i\lambda_n t}\}$ has a unique expansion of the form $\text{l.i.m.}_{N\to\infty} \sum_{-N}^{N} c_n e^{i\lambda_n t}$ with $\{c_n\}$ in l^2. Since the exponentials $e^{i\lambda_n t}$ are complete in $L^2(-\pi, \pi)$ the result follows.

Theorem 6. *Let $\{\lambda_n\}$ be a complete interpolating sequence. There exist positive numbers δ_n such that $\{\mu_n\}$ is a complete interpolating sequence whenever $|\lambda_n - \mu_n| < \delta_n$.*

Proof. Since $\{\lambda_n\}$ is interpolating there is a constant $A > 0$ such that

$$A \sum |c_n|^2 \leq \|\sum c_n e^{i\lambda_n t}\|^2_{L^2(-\pi,\pi)} \tag{18}$$

for every finite sequence $\{c_n\}$. If $\delta_n > 0$ is chosen small enough so that

$$\sum_{n=1}^{\infty} \|e^{i\lambda_n t} - e^{i\mu_n t}\|^2_{L^2(-\pi,\pi)} \leq \frac{A}{2}$$

whenever $|\lambda_n - \mu_n| < \delta_n$ $(n = 1, 2, \ldots)$, then for every finite sequence $\{c_n\}$

$$\begin{aligned}\|\sum c_n (e^{i\lambda_n t} - e^{i\mu_n t})\|^2_{L^2(-\pi,\pi)} &\leq [\sum |c_n| \, \|e^{i\lambda_n t} - e^{i\mu_n t}\|]^2 \\ &\leq [\sum |c_n|^2][\sum \|e^{i\lambda_n t} - e^{i\mu_n t}\|^2] \\ &\leq \frac{A}{2} \sum |c_n|^2.\end{aligned} \tag{19}$$

Combining (18) and (19) we get

$$\|\sum c_n (e^{i\lambda_n t} - e^{i\mu_n t})\|^2_{L^2(-\pi,\pi)} \leq \tfrac{1}{2} \|\sum c_n e^{i\lambda_n t}\|^2_{L^2(-\pi,\pi)}$$

whenever $|\lambda_n - \mu_n| < \delta_n$ $(n = 1, 2, \ldots)$. Since $\{e^{i\lambda_n t}\}$ is complete in $L^2(-\pi, \pi)$, it follows from a theorem of Boas [2, p. 469] that $\{e^{i\mu_n t}\}$ is also complete. In Theorem 3 it was shown that $\{\mu_n\}$ is interpolating whenever the λ_n are sufficiently small, whence the result follows.

4. Interpolation in E_τ^p. We use the standard notation E_τ^p to denote the space of entire functions of exponential type τ $(0 < \tau < \infty)$ which belong to $L^p(-\infty, \infty)$ on the real axis. For the properties of the spaces E_τ^p see [4]. For $0 < p < \infty$, let

$$\|f\|_p = \left[\int_{-\infty}^{\infty} |f(x)|^p dx\right]^{1/p},$$

while for $p = \infty$, let $\|f\|_\infty = \sup|f(x)|$ (x real).

Definition. A sequence $\{\lambda_n\}$ of distinct complex numbers is called an *interpolating sequence* for E_τ^p if $TE_\tau^p \supset l^p$. Here we continue to denote by T the mapping $f \to \{f(\lambda_n)\}$.

The following results are derived from Lemmas 2 and 3 in essentially the same way as Proposition 1 and Corollary 1. The proofs are therefore omitted.

Proposition 3. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TE_\tau^p \supset l^1$ $(1 \le p \le \infty)$, then $\{\lambda_n\}$ is separated.*

Corollary 2. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis. If $TE_\tau^p \supset l^1$ $(1 \le p \le \infty)$, then $TE_\tau^p \subset l^p$.*

The remainder of this section is devoted to interpolation in E_τ^p in the special cases $p = 1$ and $p = \infty$.

Theorem 7. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis and suppose that there exist functions f_n in E_μ^∞ satisfying*

$$f_n(\lambda_k) = \delta_{nk}, \qquad \|f_n\|_\infty \le M, \qquad (\text{all } n, k).$$

Then $TE_\tau^\infty = l^\infty$ whenever $\tau > \mu$.

Proof. It is well known [4, p. 82] that for every f in E_τ^∞

$$|f(x + iy)| \le \|f\|_\infty e^{\tau|y|}, \tag{20}$$

so that $TE_\tau^\infty \subset l^\infty$ for each $\tau > 0$.

Fix $\{c_n\}$ in l^∞, $\tau > \mu$ and let $\varepsilon = (\tau - \mu)/2$. We show that the function

$$f(z) = \sum_{-\infty}^{\infty} c_n f_n(z)\left[\frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)}\right]^2 \tag{21}$$

belongs to E_τ^∞. Clearly, $f(\lambda_n) = c_n$ $(-\infty < n < \infty)$. Let $\lambda_n = \alpha_n + i\beta_n$ and suppose that $|\beta_n| \le \alpha$ and $|c_n| \le N$. For $m = 0, 1, 2, \ldots$, let S_m be the set of integers n for which $m - 1 \le |\lambda_n| \le m + 2$ and T_m the set of n for which $|\lambda_n| < m - 1$ or $|\lambda_n| > m + 2$. The method of proof of Proposition 1 shows that $\{\lambda_n\}$ is separated. Since $\{\beta_n\}$ is bounded there is a constant K, independent of m, such that the number of integers in S_m is at most K. For $m \le |z| \le m + 1$, write

$$f(z) = \sum_{n \in S_m} c_n f_n(z)\left[\frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)}\right]^2 + \sum_{n \in T_m} c_n f_n(z)\left[\frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)}\right]^2.$$

Since $(\sin z)/z$ is entire of exponential type 1 and is bounded by 1 on the real axis, (20) shows that $|(\sin z)/z| \leq e^{|\mathrm{Im}\, z|}$. Therefore, setting $z = x + iy$, we have

$$
\begin{aligned}
|f(z)| &\leq NM \exp(\mu|y|) \sum_{n \in S_m} \exp(2\varepsilon|y - \beta_n|) \\
&\quad + \frac{NM \exp(\mu|y|)}{\varepsilon^2} \sum_{n \in T_m} \frac{\exp(2\varepsilon|y - \beta_n|)}{|z - \lambda_n|^2} \\
&\leq KMN \exp((\mu + 2\varepsilon)|y| + 2\varepsilon\alpha) \\
&\quad + \varepsilon^{-2} NM \exp((\mu + 2\varepsilon)|y| + 2\varepsilon\alpha) \sum_{n \in T_m} \frac{1}{|z - \lambda_n|^2}.
\end{aligned} \tag{22}
$$

We claim that the sums $\sum_{n \in T_m} |z - \lambda_n|^{-2}$ have a uniform upper bound for all $m \geq 0$ and $m \leq |z| \leq m + 1$. Since $\{\lambda_n\}$ is a separated sequence, our assertion is immediate when each λ_n is real, while in the general case the existence of an upper bound follows readily from the boundedness of Im λ_n. It follows from (22) that the series in (21) converges uniformly in each disk $|z| \leq m$ $(m = 1, 2, \ldots)$ and that, for some constant A, $|f(z)| \leq A \exp[(\mu + 2\varepsilon)|y|]$. Since $\tau = \mu + 2\varepsilon$, f belongs to E_τ^∞ and the proof is complete.

Theorem 8. *If $\{\lambda_n\}$ is a real sequence with $\lambda_{n+1} - \lambda_n \geq 1$ $(-\infty < n < \infty)$, then $TE_\tau^\infty = l^\infty$ whenever $\tau > \pi$.*

Proof. That $TE_\tau^\infty \subset l^\infty$ is clear. It follows readily from Theorem 1 that $\{\lambda_n\}$ is an interpolating sequence for E_μ^2 whenever $\mu > \pi$. Indeed, if we set $\mu_n = (\mu/\pi)\lambda_n$, then $\mu_{n+1} - \mu_n \geq \mu/\pi > 1$ and Theorem 1 shows that $\{\mu_n\}$ is an interpolating sequence for E_π^2. Therefore, given $\{c_n\} \in l^2$ there exists a function g in E_π^2 such that $g(\mu_n) = c_n$ for all n. Setting $f(z) = g((\mu/\pi)z)$ we see that f belongs to E_μ^2 and $f(\lambda_n) = c_n$ (all n), and this establishes our assertion. Let us now fix μ with $\pi < \mu < \tau$. Lemma 2 shows that there exist functions f_n in E_μ^2 for which

$$f_n(\lambda_k) = \delta_{nk}, \qquad \sup_n \|f_n\|_2 < \infty, \qquad (\text{all } n, k).$$

From (7) it follows that $|f(x)|^2 \leq (\mu/\pi)\|f\|_2^2$ for all f in E_μ^2 and all real x, so that $\sup_n \|f_n\|_\infty < \infty$. The conclusion now follows from Theorem 7.

In the same way we get the following result.

Theorem 9. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis and suppose that*

$$|\mathrm{Re}(\lambda_n) - n| \leq L < (\log 2)/\pi \qquad (-\infty < n < \infty).$$

Then $TE_\tau^\infty = l^\infty$ whenever $\tau > \pi$.

Theorem 10. *The integers are not an interpolating sequence for E_π^∞.*

Proof. We show that the sequence $\{c_n\}$ given by

$$(23)\qquad \begin{aligned} c_n &= 0, & n \le 0,\\ &= (-1)^n, & n > 0,\end{aligned}$$

cannot be interpolated. Suppose first that $\{w_n\}$ is an arbitrary sequence in l^∞ and that $w_0 = 0$. If there is an f in E_π^∞ with $f(n) = w_n$ $(-\infty < n < \infty)$, then $g(z) = f(z)/z$ is in E_π^2 and $ng(n) = w_n$. If h is any other function in E_π^2 for which $zh(z)$ belongs to E_π^∞ and $nh(n) = w_n$ $(-\infty < n < \infty)$, then $h(z) = g(z) + \alpha(\sin \pi z)/\pi z$ for some complex number α [4, p. 221]. Thus

$$f(z) = z\left[\sum_{n\neq 0} \frac{w_n}{n}\frac{\sin \pi(z-n)}{(z-n)} + \alpha\frac{\sin \pi z}{\pi z}\right]$$

is the most general function in E_π^∞ with $f(n) = w_n$ $(n \neq 0)$ and $f(0) = 0$.

A necessary condition that $zg(z)$ belong to E_π^∞ is that its derivative $zg'(z) + g(z)$ be bounded on the real axis [4, p. 206] and hence, in particular, that $ng'(n) + g(n)$ be bounded uniformly in n. Since g belongs to E_π^2, $g(n) \to 0$ as $|n| \to \infty$, so that $\{ng'(n)\}$ must be bounded.

Now, let $\{c_n\}$ be given by (23) and let

$$g(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\frac{\sin \pi(z-n)}{\pi(z-n)}.$$

We will show that $zg(z)$ is not bounded on the real axis by showing that $|ng'(n)| \to \infty$ as $n \to \infty$. By the preceding remarks the integers cannot be interpolating for E_π^∞.

We have

$$g'(z) = \sum_{n=1}^{\infty} \frac{1}{n}\frac{\pi^2(z-n)\cos \pi z - \pi \sin \pi z}{\pi^2(z-n)^2}$$

so that for $k > 0$,

$$g'(k) \sum_{n=1;n\neq k}^{\infty} \frac{1}{n}\frac{\cos \pi k}{k-n} = \sum_{n=1;n\neq k}^{\infty} \frac{1}{n}\frac{(-1)^k}{k-n}.$$

Thus

$$kg'(k) = (-1)^k \sum_{r=1;n\neq k}^{\infty} \left(\frac{1}{n} - \frac{1}{n-k}\right).$$

It is not difficult to show that

$$\sum_{n=1;n\neq k}^{\infty} \left(\frac{1}{n} - \frac{1}{n-k}\right) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) - \frac{2}{k}$$

so that $|kg'(k)| \to \infty$ as $k \to \infty$.

Theorem 11. *If* $\{\lambda_n\}$ *is a real sequence with* $\lambda_{n+1} - \lambda_n \ge 1$ $(-\infty < n < \infty)$, *then* $TE_\tau^1 = l^1$ *whenever* $\tau > \pi$.

Proof. Lemma 3 shows that $TE^1_\tau \subset l^1$. It follows just as in the proof of Theorem 8 that for $\pi < \mu < \tau$ there exist functions g_n in E^2_μ with $g_n(\lambda_k) = \delta_{nk}$ and $\sup_n \|g_n\|_2 < \infty$. If we set $\varepsilon = \tau - \mu$ and let

$$f_n(z) = g_n(z)(\sin \varepsilon(z - \lambda_n))/\varepsilon(z - \lambda_n),$$

then $f_n \in E^1_\tau$ and $f_n(\lambda_k) = \delta_{nk}$. Hölder's inequality shows that

$$\|f_n\|_1 \le \|g_n\|_2 \left\| \frac{\sin \varepsilon(z - \lambda_n)}{\varepsilon(z - \lambda_n)} \right\|_2$$

and it follows that $\sup_n \|f_n\|_1 < \infty$.

Now, choose $\{c_n\}$ in l^1 and set

$$f(z) = \sum_{-\infty}^{\infty} c_n f_n(z). \tag{24}$$

Since $\sum \|c_n f_n\|_1 < \infty$, f belongs to E^1_τ, and Lemma 3 implies that the convergence in (24) is uniform in each horizontal strip. Therefore, $f(\lambda_n) = c_n$ $(-\infty < n < \infty)$ and the proof is complete.

It is easy to see that this result is best possible, in the sense that τ cannot always be taken equal to π. Indeed, the integers are not an interpolating sequence for E^1_π for the trivial reason that the nonzero integers are a set of uniqueness. However, we have the following stronger result.

Theorem 12. *The nonzero integers are not an interpolating sequence for E^1_τ.*

Proof. Lemma 3 shows that point evaluations are continuous linear functionals on E^1_τ. By Lemma 2 it is enough to show that the unit ball of l^1 cannot be interpolated in a uniformly bounded way. Since

$$f_n(z) = n(\sin \pi(z - n))/\pi z(z - n) \qquad (n \neq 0)$$

is the unique function in E^1_π with the property that $f_n(k) = \delta_{nk}$, it is sufficient to show that $\|f_n\|_1 \to \infty$ as $n \to \infty$. For $n > 0$,

$$\begin{aligned}
\|f_n\|_1 = \int_{-\infty}^{\infty} |f_n(x)|\,dx &> \frac{n}{\pi}\int_1^{\infty} \left|\frac{\sin \pi x}{x(x+n)}\right| dx \\
&> \frac{n}{\pi} \sum_{k=1}^{\infty} \int_{k+1/4}^{k+3/4} \left|\frac{\sin \pi x}{x(x+n)}\right| dx \\
&> \frac{\sqrt{2}}{2\pi} \sum_{k=1}^{\infty} \int_{k+1/4}^{k+3/4} \left(\frac{1}{x} - \frac{1}{x+n}\right) dx \\
&= \frac{\sqrt{2}}{2\pi} \log \prod_{k=1}^{\infty} \left[\frac{k+3/4}{k+n+3/4}\,\frac{k+n+1/4}{k+1/4}\right].
\end{aligned}$$

Using the relation $\Gamma(x + 1) = x\Gamma(x)$ it easily follows that the infinite product above is equal to

$$\lim_{N\to\infty}\left[\frac{(1+3/4)(2+3/4)\cdots(N+3/4)}{(1+n+3/4)(2+n+3/4)\cdots(N+n+3/4)}\right]$$
$$\cdot\left[\frac{(1+n+1/4)\cdots(N+n+1/4)}{(1+1/4)\cdots(N+1/4)}\right]$$
$$=\left[\frac{1+3/4}{1+1/4}\right]\left[\frac{2+3/4}{2+1/4}\right]\cdots\left[\frac{n+3/4}{n+1/4}\right]$$
$$=\left[\frac{\Gamma(1+1/4)}{\Gamma(1+3/4)}\right]\left[\frac{\Gamma(n+1+3/4)}{\Gamma(n+1+1/4)}\right].$$

From the estimate $\Gamma(x+1)\sim(2\pi)^{1/2}x^{x+1/2}e^{-x}$ (as $x\to\infty$) we conclude that $\Gamma(n+1+3/4)/\Gamma(n+1+1/4)\to\infty$ (as $n\to\infty$), and the proof is complete.

The proof of the next theorem is similar to that of Theorem 11 and is therefore omitted.

Theorem 13. *Let $\{\lambda_n\}$ be a sequence of points lying in a strip parallel to the real axis and suppose that*

$$|\mathrm{Re}(\lambda_n)-n|\le L<(\log 2)/\pi \qquad (-\infty<n<\infty).$$

Then $TE_\tau^1 = l^1$ whenever $\tau>\pi$.

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